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ON VARIOUS CASES OF INSTABILITY FOR ELASTIC NONCONSERVATIVE SYSTEMS WITH DAMPING

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Abstract—We investigate the stability of a mechanical system with viscous damping when subjected to nonconservative forces. The spectrum corresponding to the eigenvalue problem is proved to have a symmetrical property, and therefore the zero frequency is always at least double. Here, we consider the following important cases: where two linearly independent eigenfunctions correspond to zero frequency, and where only one eigenfunction corresponds to zero frequency (bimodal and unimodal cases). The full description of all possible variants of the system behaviour in the neighbourhood of critical points is obtained. Examples are considered.

1. INTRODUCTION

The theory of stability of elastic systems mostly deals with conservative systems, i.e. with the systems for which the theorem of energy conservation holds. However, there are problems of practical importance where one should consider nonconservative elastic systems. These problems are connected, for example, with the torsion of shafts, pipes conducting media, panels exposed to the flow of fluids or gases, etc. Nicolai (1927) and Ziegler (1951, 1952) were the first to attract the attention of scientists and engineers to the nontrivial behaviour of such systems. A significant contribution to their study was made by Bolotin (1963), Ziegler (1968) and Leipholz (1980a, b). These books are the most well known in this field. When considering the behaviour of nonconservative systems one should distinguish between two types of instability, the dynamic instability (flutter) and the static instability (divergence). For the nonconservative systems, usual extremum principles like Rayleigh's principle, do not take place. However, as shown by Leipholz (1974, 1980a, b) and Prasad and Herrmann (1969, 1972), consideration of the adjoint problem allows us to use a stationarity principle.

Ziegler (1952) discovered that small viscous damping can give rise to instability in a nonconservative elastic system [see also Bolotin and Zinzher (1969), Herrmann and Jong (1965) Nemat-Nasser and Herrmann (1966), Andreichikov and Yudovich (1974), Denisov and Novikov (1978), and Miloslavsky (1986)]. The examples of viscous damping forces, that stabilize or destabilize nonconservative systems were described in Banichuk *et al.* (1990), Banichuk and Bratus (1990, 1992), and Bratus (1991). In this investigation an important part is played by both adjoint problems and spectrum perturbation theory for nonself-adjoint operators and matrices [see Vishik and Lyusternik (1960)]. Note that the most interesting cases are connected with the appearance of multiple eigenvalues (frequencies).

The purpose of this paper is to investigate the behaviour of a nonconservative system with damping in the neighbourhoods of the critical points. We show that the spectrum corresponding to the eigenvalue problem has a symmetrical property, therefore the zero frequency is always at least double. Two important cases are considered, when the eigenvalue problem has two linearly independent eigenfunctions (bimodal case) and a unique eigenfunction (unimodal case). By using the perturbation methods we have described completely the system's behaviour in all cases, when the loading parameter steps over the critical value. As an example, we apply our analysis to the stability problem for a pipe containing flowing fluid.

2. STATEMENT OF THE PROBLEM

Free oscillations of elastic bodies are described by the following equation with homogeneous boundary conditions [see for example Bolotin (1963), Ziegler (1968), and Leipholz (1980a, b)]:

$$A(p)u(x,t)+B(p)\frac{\partial}{\partial t}u(x,t)+\frac{\partial^2 u}{\partial t^2}(x,t)=0, \quad 0<x<l, t>0, \tag{1}$$

$$(C_{1j}(p)u)_{x=0}=0, \quad (C_{2j}(p)u)_{x=l}=0, \quad j=1,2,\dots,m. \tag{2}$$

Here, $u(x, t)$ is the displacement, $A(p)$, $B(p)$ are linear differential operators (with respect to the variable x) with real coefficients depending analytically on the real loading parameter $p \in \mathbb{R}$,

$$A(p)=\sum_{j=0}^{2m} a_j(p)\frac{\partial^j}{\partial x^j}, \quad B(p)=\sum_{j=0}^{r} b_j(p)\frac{\partial^j}{\partial x^j}, \quad r \leqslant 2m,$$

$C_{1j}(p)$, $C_{2j}(p)$ ($j = 1, 2, \dots, m$) are linear differential operators of the orders not more than $2m$. The linear differential operators C_{1j} and C_{2j}, $j = 1, 2, \dots, m$, depend on p, like $A(p)$ and $B(p)$. We assume in what follows that the boundary value problem (1), (2) is of the Sturm type [see Naimark (1969)].

The eqns (1) and (2) describe a wide variety of elastic systems (beams, columns, arches, plates, shells). The structure of the operator $A(p)$ depends on the rigidity properties of the elastic body as well as on the mode of its loading. The form of the operator $B(p)$ is defined by the influence of damping forces. The boundary condition (2) depends on how the body is clamped and loaded on its boundaries. In problems related to mechanics, the operator $A(p)$ is represented, as a rule, in the form $N+pS$, where N and S are linear differential operators with real coefficients, the loading parameter p being included linearly. If the system is conservative, then the operators N and S are self-adjoint operators with respect to boundary conditions (2), N being positive definite and S being negative definite. In the case under consideration, the operator $A(p)$ is not necessarily self-adjoint on the set of functions that satisfy boundary condition (2).

When studying the stability of systems (1), (2) we assume that the solution can be obtained in the form of a Fourier series in the variable $t \in [0, T]$. Particular solutions to this system are sought in the form

$$u(x,t)=v(x)\exp[\mathrm{i}\omega t],$$

where $v(x)$ is the amplitude function, square integrable on the segment $[0, l]$ and satisfying boundary conditions (2), and i is an imaginary unit, ω is the frequency.

By substituting the last expression into eqn (1) we obtain a relation between the amplitude function $v(x)$ and the frequency ω in the form of the eigenvalue problem with a parameter p:

$$L(p)v(x)=A(p)v(x)+\mathrm{i}\omega B(p)v(x)-\omega^2 v(x)=0. \tag{3}$$

In what follows, we assume that the eigenvalue boundary value problem (3), (2) has a discrete spectrum, i.e. for each value of the loading parameter p there exists a countable set of solutions $\{\omega_j(p)\}_{j=1}^{\infty}$.

Definition 2.1. The value p_0 of the loading parameter is called a critical value if at least one of the eigenvalues $\omega_j(p)$, $j = 1, 2, \dots,$ of problem (3), (2) satisfies the condition

$$\operatorname{Im}\omega_j(p_0)=0,$$

and in a small neighbourhood of the point p_0 the inequality

$$\operatorname{Im}\omega_j(p) \neq 0$$

is fulfilled, for $p \neq p_0$.

If Re $\omega_j(p_0) = 0$ and Im $\omega_j(p) < 0$ for $p > p_0$, we say that the system loses its stability in a static way (divergence), and if Re $\omega_j(p_0) \neq 0$, then the dynamical form of instability takes place (flutter).

We will study the behaviour of the system (1), (2) in the static case (divergence) in the neighbourhood of critical points.

3. SYMMETRICAL PROPERTY OF FREQUENCIES

Theorem 3.1. *Let* $\omega_j = \tau_j + i\sigma_j$, $j = 1, 2, 3, \ldots$ *be the eigenvalues of the boundary problem* (3), (2) *for* $p \in \mathbb{R}$. *Then this problem also has the eigenvalues* $\omega'_j = -\tau_j + i\sigma_j$, *i.e. the spectrum is symmetrical with respect to the imaginary axis* $\operatorname{Im}\omega_j$.

Proof. Consider the equation

$$A(p)w(x) - \omega^2 w(x) = 0 \tag{4}$$

together with boundary condition (2) for the function $w(x)$. From the assumption on the character of the spectrum, using the results of Michailov (1962) we conclude that the system of eigenfunctions and their joint functions, i.e. $\{w_j(x)\}_{j=1}^{\infty}$, form the basis, in the sense of Riesz in the space of square integrable functions on the segment $[0, l]$. We will seek the approximate solution of the boundary value problem (1), (2) in the form of finite expansion in the system $\{w_j(x)\}_{j=1}^{n}$, i.e.

$$v^n(x) = \sum_{j=1}^{n} c_j w_j(x),$$

where the coefficients c_j are complex numbers, in the general case.

Denote by ω_n the corresponding approximation for the frequencies ω of the problem (3), (2). Substituting $v^n(x)$ into (4) and multiplying the resultant equation scalarly by the functions $w_j(x)$, $j = 1, 2, \ldots, n$, we get the following linear system of equations for the coefficients c_j:

$$\sum_{j=1}^{n} c_j(\delta_{sj} + i\omega_n \chi_{sj} - \omega_n^2 \gamma_{sj}) = 0, \quad s = 1, 2, \ldots, n, \tag{5}$$

where

$$\delta_{sj} = (A(p)w_j, w_s), \quad x_{sj} = (B(p)w_j, w_s), \quad \gamma_{sj} = (w_s, w_j).$$

Here and below, the brackets denote the scalar product in the space of square integrable functions. Note that the values δ_{sj}, χ_{sj} and γ_{sj} are real, since the functions $w_j(x)$ are real.

For solvability of the system (5) it is necessary that the following determinant is equal to zero:

$$\Delta_n = \det[\delta_{sj} + (i\tau_n - \sigma_n)x_{sj} - ((\tau_n^2 - \sigma_n^2) + 2i\tau_n\sigma_n)\gamma_{sj}] = 0, \tag{6}$$

here $\omega_n = \tau_n + i\sigma_n$.

Since all terms in eqn (6) are real, except for $i\tau_n$ and $i\tau_n\sigma_n$, the following equality holds:

$$\Delta_n = P_n^1(\tau_n^2, \sigma_n, p) + i\tau_n P_n^2(\tau_n^2, \sigma_n, p),$$

where $P_n^k(z, \sigma_n, p)$, $k = 1, 2$ are polynomials in z and σ_n with the coefficients depending on p. If for some $p \in \mathbb{R}$ eqn (6) has a solution $\omega_n = \tau_n + i\sigma_n$, ($\tau_n \neq 0$) then for the same p,

$$P_n^k(\tau_n^2, \sigma_n, p) = 0, \quad k = 1, 2.$$

By virtue of evenness of the polynomials P_n^k with respect to $z = \tau_n^2$, eqn (6) with the same p and σ_n also has the solution $\omega_n' = -\tau_n + i\sigma_n$. Since the solution $v^n(x)$ converges as $n \to \infty$ to the solution of the boundary value problem (3), (2) in the norm of the space of square integrable functions [see Strang and Fix (1973)] and the limit of a sequence of even functions is an even function we get the assertion of Theorem 3.1.

Corollary 3.1. Suppose that for some $p_0 \in \mathbb{R}$ there exists the zero eigenvalue $\omega(p_0) = 0$ of the problem (3), (2) and the problem

$$A(p_0)v(x) = 0, \tag{7}$$

with the boundary condition (2), has a unique solution. Also suppose that for a sufficiently small neighbourhood of the point p_0 the inequality Re $\omega(p) \neq 0$ is fulfilled, where $\omega(p)$ is such an eigenvalue of the problem (3), (2) that $\omega(p_0) = 0$. Then the zero eigenvalue $\omega(p_0) = 0$ is a double eigenvalue of the problem (3), (2).

Proof. For sufficiently small $p = p_0 \neq 0$ we have $\omega(p) = \tau(p) + i\sigma(p)$, where $\tau(p) \neq 0$ for $p \neq p_0$. On the other hand, for the same value of the parameter p the problem (3), (2) has symmetrical eigenvalues $\omega'(p) = -\tau(p) + i\sigma(p)$. By virtue of the continuity of the eigenvalues with respect to p we get $\tau(p) \to 0$ and $\sigma(p) \to 0$ as $p \to p_0$. Therefore, the zero eigenvalue $\omega(p_0)$ is a double eigenvalue of the problem (3), (2).

4. SERIES EXPANSIONS

Consider the increment of the parameter p at the point p_0 such that $\omega(p_0) = 0$: $p_\alpha = p_0 + \alpha$, where α is a sufficiently small positive number. All eigenvalues and eigenfunctions of the boundary problem (3), (2) will have some increments too. Vishik and Lyusternik (1960) have shown that if s is the order of the multiple eigenvalue $\omega(p_0)$ and k is the number of linearly independent eigenfunctions that correspond to this eigenvalue, then the expansions of the eigenvalues $\omega(p_\alpha)$ and corresponding eigenfunction $v_\alpha(x)$ are to be made in fractional powers of $\alpha^{k/s}$. Therefore, if the assumptions of Corollary 3.1 are fulfilled then the expansions of the value $\omega(p_\alpha)$ and corresponding eigenfunction $v_\alpha(x)$ are made in powers of $\alpha^{1/2}$:

$$\omega(p_\alpha) = \omega_\alpha = \alpha^{1/2}\mu + \alpha\eta + O(\alpha^{3/2}), \tag{8}$$

$$v_\alpha(x) = v_0(x) + \alpha^{1/2}v_1(x) + \alpha v_2(x) + O(\alpha^{3/2}). \tag{9}$$

Here, $v_0(x)$ is the solution of problem (7) with boundary conditions (2) and μ, η are some coefficients.

Without loss of generality we consider that the eigenfunction $v_\alpha(x)$ is normalized in the following way:

$$(v_\alpha(x), v_0(x)) = 1. \tag{10}$$

The next step is to arrange the coefficients for each power of α into groups. Since the equality must be satisfied for arbitrary values of α, the coefficients of powers of α must be equal to zero.

Introducing the notation

$$A^0 = A(p_0), \quad B^0 = B(p_0), \quad A_p^0 = \left(\frac{d}{d\alpha}A(p+\alpha)\right)_{\alpha=0} \tag{11}$$

we get:

$$A^0 v_0(x) = 0, \tag{12a}$$

$$A^0 v_1(x) = -\mathrm{i}\mu B^0 v_0(x), \tag{12b}$$

$$A^0 v_2(x) = -A_p^0 v_0(x) - \mathrm{i}\mu B^0 v_1(x) - \mathrm{i}\eta B^0 v_0(x) + \mu^2 v_0(x). \tag{12c}$$

If the boundary value problem (12a), (2) has two linearly independent solutions $v_0^1(x)$ and $v_0^2(x)$, the expansion for $\omega(p_\alpha)$ and $v_\alpha(x)$ can be made in integer powers of α. For example, this takes place if the order of the multiple zero eigenvalue is exactly two [see Vishik and Lyusternik (1960) and Kato (1966)]. In any case we can conclude that the expansion contains powers of $\alpha^{2/s}$, where s is the order of the multiple zero eigenvalue. Let us consider the first case ($s = 2$), then we get the following expansions:

$$\omega(p_\alpha) = \omega_\alpha = \alpha\mu + \alpha^2\eta + O(\alpha^3), \tag{13}$$

$$v_\alpha(x) = v_0(x) + \alpha v_1(x) + \alpha^2 v_2(x) + O(\alpha^3), \tag{14}$$

where

$$v_0(x) = c_1 v_0^1(x) + c_2 v_0^2(x), \quad |c_1| + |c_2| \neq 0, \tag{15}$$

c_1, c_2 are constants.

If we arrange coefficients for each power of α into groups we get eqn (12a) and the following equation:

$$A^0 v_1(x) = -A_p^0 v_0(x) - \mathrm{i}\mu B^0 v_0(x). \tag{16}$$

5. UNIMODAL DOUBLE EIGENVALUE

Consider the boundary value problem, adjoint to the boundary value problem (3), (2):

$$A^{\mathrm{T}}(p)z(x) - \mathrm{i}\omega B^{\mathrm{T}}(p)z(x) - \omega^2 z(x) = 0, \tag{17}$$

$$(C_{1j}^{\mathrm{T}}(p)z)_{x=0}, \quad (C_2^{\mathrm{T}}(p)z)_{x=1} = 0, \quad j = 1, 2, \ldots, m. \tag{18}$$

Let us denote the solution to the adjoint problem for critical value p_0 by $z_0(x)$. As mentioned above, this case corresponds to the zero eigenvalue $\omega(p_0) = 0$. We know that the following equality holds [see Vishik and Lyusternik (1960)]:

$$(v_0(x), z_0(x)) = 0, \tag{19}$$

where $v_0(x)$ is the solution of the boundary value problem (12a), (2). Equation (13) has a solution if and only if its right-hand side is orthogonal to the function $z_0(x)$. Then we get

$$\mu(B^0 v_0, z_0) = 0. \tag{20}$$

If $(B^0 v_0, z_0) = 0$, then eqn (12b) is always solvable. Therefore there exists a real operator $G(p_0) = G^0$ (Green's function) such that

$$v_1(x) = -\mathrm{i}\mu G^0 B^0 v_0(x). \tag{21}$$

Denote

$$M = (A_p^0 v_0, z_0)/(B^0 v_0, z_0), \tag{22a}$$

$$N = -(A_p^0 v_0, z_0)/(B^0 G^0 B^0 v_0, z_0). \tag{22b}$$

Theorem 5.1. *Suppose that for* $p = p_0$, *only one eigenfunction*, $v_0(x)$, *corresponds to the double zero eigenvalue of the boundary value problem* (3), (2), *and let* $z_0(x)$ *be the eigenfunction of the adjoint boundary value problem* (17), (18).

If $(B^0v_0, z_0) \neq 0$ and $M < 0$, then the system (1), (2), being asymptotically stable, becomes unstable when the parameter p exceeds the critical value p_0 and the loss of stability has a static character (divergence). If $M > 0$, then the system (1), (2), being statically unstable, becomes asymptotically stable. For $M = 0$ the following representation takes place:

$$\omega_\alpha = 0(\alpha^{3/2}).$$

If $(B^0v_0, z_0) = 0$ and $N < 0$, then for sufficiently small $p - p_0 > 0$ the system (1), (2) becomes unstable in a statical manner. If $N \geqslant 0$, then the double zero eigenvalue splits into two different eigenvalues so that

$$\omega_\alpha = \pm\alpha^{1/2}N^{1/2} + O(\alpha).$$

Proof. If $(B^0v_0, z_0) \neq 0$, then eqn (20) implies that $\mu = 0$. Using eqn (12b) and condition (10) we get $v_1(x) = 0$. Equation (12c) has the following form:

$$A^0v_2(x) = -A_p^0v_0(x) - i\eta B^0v_0(x).$$

The condition of solvability results in the following equality:

$$(A_p^0v_0, z_0) + i\eta(B^0v_0, z_0) = 0.$$

Taking into account the notations (20) and assuming $\eta = iM$ we get the first assertion of Theorem 5.1. Here we have also used the fact that the terms A_p^0 change their signs, as the parameter changes its sign. Therefore, the case of $M < 0$ (>0) turns into the case of $M > 0$ (<0), as the parameter p passes through the critical value p_0.

If $(B^0v_0, z_0) = 0$ then the equality (21) takes place. Substituting it into eqn (14) and multiplying the resultant relationship scalarly by the eigenfunction $z_0(x)$ of the adjoint boundary value problem (18), (19) we get $\mu^2 = N$, where N is defined by (22b). The second part of the assertion of Theorem 5.1 is proved analogously.

Corollary 5.1. The results of Theorem 5.1. hold if for $p = p_0$ the boundary problem (12a), (2) is conservative and therefore, the operator A^0 is a self-adjoint. The function $z_0(x)$ must be replaced with the function $v_0(x)$, and eqn (22b) is obtained in the form

$$N = (A_p^0v_0, v_0)/((B^0G^0B^0v_0, v_0) - (v_0, v_0)), \qquad (23)$$

since in this case the equality (19) is not valid [see Vishik and Lyusternik (1960)].

The results are illustrated in Figs 1 and 2. The arrows show the displacement of the eigenvalues as the loading parameter passes through the critical value p_0.

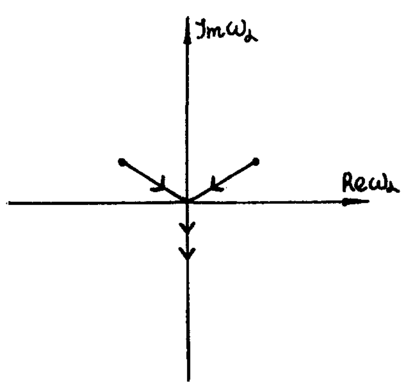

Fig. 1. The case $(B^0v_0, z_0) \neq 0$, $M < 0$.

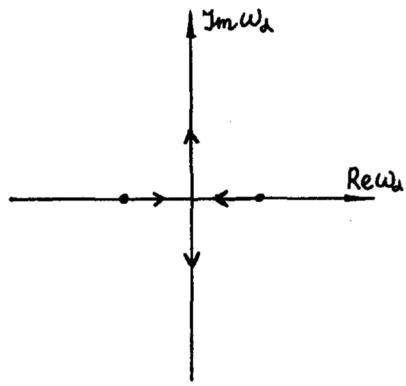

Fig. 2. The case $(B^0 v_0, z_0) = 0, N < 0$.

Example 5.1. The stability problem for the pipe through which fluid is flowing leads to the following boundary value problem [see for example Panovko and Gubanova (1979)]:

$$\frac{d^4 v(x)}{dx^4} + p^2 \frac{d^2 v(x)}{dx^2} + 2i\beta p\omega \frac{dv}{dx} - \omega^2 v(x) = 0, \quad 0 < x < 1, \tag{24a}$$

$$v(0) = \frac{d^2 v}{dx^2}(0) = 0, \quad v(1) = \frac{d^2 v}{dx^2}(1) = 0, \tag{24b}$$

where p is the flow velocity, $\beta > 0$ is a constant, ω is the frequency of bending oscillation of the pipe. According to the notation of Sections 2, 3 and 4 we have

$$A(p)v(x) = v^{(IV)}(x) + p^2 v''(x), \quad B(p)v(x) = 2\beta p v'(x),$$
$$A_p(p)v(x) = 2pv''(x).$$

If $p_n = \pi n$, $n = 1, 2, \ldots$, we get $\omega_n = \omega(p_n) = 0$, $v_0^n(x) = \sqrt{2}\sin \pi n x$. For these values of p_n and ω_n, the boundary value problem (24a), (24b) is self-adjoint. From Theorem 3.1 we conclude that for critical values $p_n = \pi n$ the zero eigenvalues are double. Since there exists only one eigenfunction, we have a unimodal case here.

It is easy to verify that

$$(B(p_n)v_0^n, v_0^n) = 2\beta(\pi n)^2 \int_0^1 \sin \pi n x \cos \pi n x \, dx = 0.$$

Therefore, $\mu^2 = N$, where N is defined by eqn (23). Let us find the function $v_1^n(x)$ satisfying eqn (12b). To do this we have to solve the following boundary value problem:

$$\frac{d^4}{dx^4} v_1^n(x) + p_n^2 \frac{d^2}{dx^2} v_1^n(x) = -2i\mu\beta\pi^3 n^2 \cos \pi n x,$$

$$v_1^n(0) = \frac{d^2}{dx^2} v_1^n(0) = 0, \quad v_1^n(1) = \frac{d^2}{dx^2} v_1^n(1) = 0.$$

The solution has the form

$$v_1^n(x) = i\sqrt{2}\frac{\mu\beta}{n}\left(x \sin \pi n x + \frac{2}{\pi x} C_n x + \frac{2}{\pi x} \operatorname{cox} \pi n x - \frac{2}{\pi n}\right),$$

where $n = 1, 2, \ldots$, $c_n = 0$ for $n = 2k$, and $c_n = 1$ for $n = 2k-1$, $k = 1, 2, \ldots$.

Now we write down eqn (12c):

$$\frac{d^4}{dx^4}v_2^n(x)+p_n^2\frac{d^2}{dx^2}v_2^n(x)+2p_n\frac{d^2v_0^n}{dx^2}+2i\mu\beta p_n\frac{dv_1^n}{dx}(x)-\mu^2v_0^n(x)=0,$$

with the same boundary condition as before. Multiplying this equation by the function $v_0^n(x)=\sqrt{2}\sin\pi nx$ and integrating the resultant equation from 0 to 1 we get the following formula for N from eqn (23):

$$N=\frac{2(\pi n)^3}{\beta^2\pi\left(3-\frac{8}{(\pi n)^2}C_n^2\right)},\quad n=1,2,\ldots$$

Note that for any $n=1,2,\ldots$ the values $3-8C_n^2/(\pi n)^2$ are positive. For $n=2k$ and $\beta<(3\pi)^{-1/2}$ as well as for $n=2k-1$ and $\beta<\pi n(3\pi^2n^2-8)^{-1}$, the zero eigenvalue for $p>\pi n$ splits into two different eigenvalues, one of which has a negative imaginary part (divergence). In other cases ($\beta>\pi n(3\pi^2n^2-8)^{-1}$), the zero eigenvalue splits into two eigenvalues, both of them being real in the first approximation in terms of powers of α. In this case we say that conditional stabilization takes place. To investigate the exact behaviour of the system an additional treatment is needed involving the terms $\alpha^{3/2}$.

In particular, for $p_1=\pi$ and $\beta<0.67582$, the static type of loss of stability takes place. If $\beta>0.67582$, then for $p>\pi$ the stabilization takes place in the conditional sense.

6. BIMODAL DOUBLE EIGENVALUE

Now consider the case of two linearly independent eigenfunctions $v_0^1(x)$ and $v_0^2(x)$ corresponding to the double zero eigenvalue. Let $z_0^1(x)$ and $z_0^2(x)$ be the corresponding eigenfunctions of the adjoint problem (17), (18). Without loss of generality we can assume that

$$(y_0^i,z_0^j)=\delta_{ij},$$

where δ_{ij} is Kronecker's delta.

Let us introduce the following notation:

$$\alpha_{ij}=(A_py_0^i,z_0^j),\quad \beta_{ij}=(B^0y_0^i,z_0^j),\quad j=1,2, \tag{25a}$$

$$a_1=\beta_{11}\beta_{22}-\beta_{21}\beta_{12},\quad a_2=\beta_{22}\alpha_{11}+\beta_{11}\alpha_{22}-\beta_{12}\alpha_2-\beta_{21}\alpha_{12},$$
$$a_3=\alpha_{11}\alpha_{22}-\alpha_{12}\alpha'_{21}. \tag{25b}$$

Theorem 6.1. *Suppose that for* $p=p_0$, *two linearly independent eigenfunctions*, $y_0^1(x)$ *and* $y_0^2(x)$, *correspond to the double zero eigenvalue of the boundary problem* (3), (2) *and let* $z_0^1(x)$ *and* $z_0^2(x)$ *be the respective eigenfunctions of the adjoint problem* (17), (18).

If the inequalities

$$a_1a_2<0,\quad a_3a_1<0 \tag{26}$$

are fulfilled, then for sufficiently small $p-p_0>0$ the system (1), (2) is asymptotically stable.

If the inequalities (26) are not fulfilled, then for the above-mentioned value of the parameter p, the dynamical destabilization (flutter) occurs, when $4a_1a_3+a_2^2<0$, and static destabilization (divergence), takes place when $4a_1a_3+a_2^2\geqslant 0$.

Proof. Consider eqn (16). Multiplying this equation by the eigenfunctions $z_0^1(x)$ and $z_0^2(x)$ of the adjoint boundary value problem (17), (18) and then integrating the result from

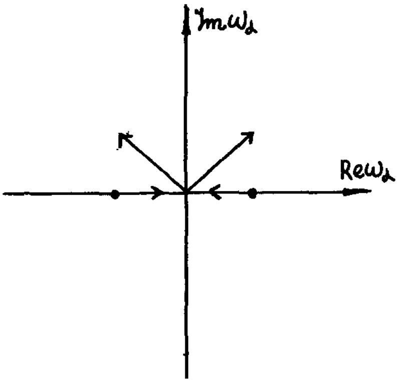

Fig. 3. The case $a_1a_2 < 0$, $a_3a_1 < 0$.

0 to l and using the notation of (25a), (25b) we get the system of linear homogeneous equations with respect to the constants c_1 and c_2 from eqn (15):

$$c_1(\alpha_{11}+i\mu\beta_{11})+c_2(\alpha_{21}+i\mu\beta_{21}) = 0,$$
$$c_1(\alpha_{12}+i\mu\beta_{12})+c_2(\alpha_{22}+i\mu\beta_{22}) = 0.$$

For the nontrivial solution to this system to exist, it is necessary that the determinant is equal to zero. This condition gives the following equation for μ from expansion (13):

$$\mu^2a_1+i\mu a_2+a_3 = 0, \tag{27}$$

where a_1, a_2, a_3 are defined by eqn (25b). Assuming $\lambda = i\mu$ and applying the Routh–Hurwitz criterion we have the situation where, if the conditions (26) are fulfilled then imaginary parts of the roots of eqn (27) will be strictly positive. Therefore for sufficiently small $p-p_0 > 0$ the initial system (1), (2) is asymptotically stable. If the inequalities are not valid and $4a_1a_3+a_2^2 < 0$ the roots of eqn (27) have nonzero real parts and their imaginary parts are negative. This means that the character of destabilization is dynamical. For $4a_1a_3+a_2^2 \geqslant 0$ the destabilization has a static character.

This reasoning can be illustrated by Figs 3 and 4.

Remark 6.1. If $a_1 = a_2 = 0$ and $a_3 \neq 0$, then eqn (27) is not valid. This means that the expansions (13), (14) do not hold either. In this case the order of the multiple zero eigenvalue can be at least equal to four, and, therefore, expansions (8), (9) and the eqns (12a, b, c) take place.

Example 6.1. Consider the stabilization problem for a pipe that lies on an elastic support and carries flowing fluid. The corresponding boundary value problem is given by

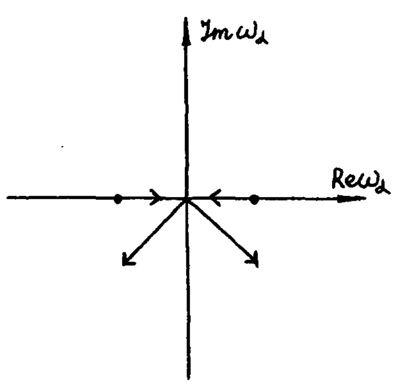

Fig. 4. The case $4a_1a_3-a_2^2 < 0$.

$$\frac{d^4v}{dx^4}(x)+p^2\frac{d^2v}{dx^2}(x)+k^2v(x)+2i\beta p\omega\frac{dv}{dx}(x)-\omega^2v(x) = 0.$$

$$0 < x < l \to 1, \tag{28a}$$

$$v(0) = \frac{d^2v}{dx^2}(0) = 0, \quad v(1) = \frac{d^2v}{dx^2}(1) = 0, \tag{28b}$$

where p, β are defined in eqn (24a), k^2 is a constant.

If $k^2 = 4\pi^4$, then for $p_0^2 = 5\pi^2$ we have $\omega(p_0) = 0$. The corresponding eigenfunctions are $v_0^1(x) = \sqrt{2}\sin \pi x$ and $v_0^2(x) = \sqrt{2}\sin 2\pi x$. The boundary value problem (28a, b) is self-adjoint for the indicated p_0 and $\omega(p_0) = 0$. Therefore $z_0^i(x) = y_0^i(x)$, $i = 1, 2$. Using the notation (25a) and (25b) we get

$$\alpha_{11} = -2\sqrt{5}\pi^3, \quad \alpha_{12} = \alpha_{21} = 0, \quad \alpha_{22} = -8\pi^3\sqrt{5},$$

$$\beta_{11} = 0, \quad \beta_{12} = 8\sqrt{5}\beta/3, \quad \beta_{21} = -8\sqrt{5}\beta/3, \quad \beta_{22} = 0.$$

Therefore,

$$a_1 = 320p^2\beta^2/9, \quad a_2 = 0, \quad a_3 = 80\pi^6.$$

The inequalities (26) are not valid. Since in this case $4a_1a_3+a_2^2 > 0$, the static destabilization takes place for any $\beta > 0$ and sufficiently small $p^2-5\pi^2 > 0$.

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